FINAL GRADUATION PROJECT MONOGRAPH

Maximum-Likelihood and Moments-Based Estimators for the α-$\mathcal{F}$ Composite Distribution

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Graduate Course in Electrical Engineering

DEPARTMENT OF ELECTRICAL ENGINEERING

UNIVERSITY OF BRASÍLIA
FACULTY OF TECHNOLOGY

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Final graduation project monograph submitted to the Department of Electrical Engineering as partial requirement for the degree of Bacharel in Electrical Engineering

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CATALOG SHEET

BRAGA, FELIPE RODRIGUES CARDOSO
VIEIRA, GUILHERME MACIEL NEVES
Maximum-Likelihood and Moments-Based Estimators for the α-$\mathcal{F}$ Composite Distribution [Federal District] 2023.
xvi, 17 p., 210 x 297 mm (ENE/FT/UnB, Bacharel, Electrical Engineering, 2023).
Final graduation project monograph - University of Brasília, Faculty of Technology.
Department of Electrical Engineering

1. Maximum-Likelihood	2. Estimators
3. α-$\mathcal{F}$ Composite Distribution	4. Method of Moments
I. ENE/FT/UnB	II. Title (series)

BIBLIOGRAPHICAL REFERENCE

BRAGA, F. R. C., VIEIRA, G. M. V. (2023). *Maximum-Likelihood and Moments-Based Estimators for the α-$\mathcal{F}$ Composite Distribution*. Final graduation project monograph, Department of Electrical Engineering, University of Brasília, Brasília, DF, DF, 17 p.

ASSIGNMENT OF RIGHTS

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TITLE: Maximum-Likelihood and Moments-Based Estimators for the α-$\mathcal{F}$ Composite Distribution
DEGREE: Bacharel in Electrical Engineering
YEAR: 2023

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ABSTRACT

Estimators for the α-$\mathcal{F}$ composite fading distribution are presented in this work, in which the maximum-likelihood (ML) method is used to estimate the parameters of the distribution under analysis and the method of moments (MoM) is adopted to estimate the signal-to-noise (SNR). In our study, an orthogonal frequency division multiplexing (OFDM) signal is considered. The performance of the new estimators are examined and several mean and variance curves are shown considering different channel parameters and SNR range. To the best of the authors' knowledge, this is the first work that address the estimation considering the α-$\mathcal{F}$ fading channels.

Keywords: α-$\mathcal{F}$ distribution, estimators, ML, MoM, OFDM signals.

SUMMARY

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List of Acronyms and Symbols

Acronyms

5G	*Fifth Generation*
ML	*Maximum Likelihood*
MoM	*Method of Moments*
NMSE	*Normalized Mean Square Error*
OFDM	*Orthogonal Frequency Division Multiplexing*
PDF	*Probability Density Function*
QAM	*Quadrature Amplitude Modulations*
SNR	*Signal-to-Noise Ratio*
RV	*Random Variable*

Symbols

$r[n]$	*Received Signal*
$s[n]$	*Transmitted signal*
$h[n]$	*Fading Coefficient*
$w[n]$	*Additive White Gaussian Noise*
$f_{\lvert\cdot\rvert}(\cdot)$	*PDF of* $\lvert\cdot\rvert$ *RV*
α	*Non-linearity channel parameter*
μ	*Multipath fading parameter*
m_s	*Shadowing fading parameter*
$\mathbb{E}\left[\lvert\cdot\rvert^k\right]$	*k-th moment of* $\lvert\cdot\rvert$
σ_H	*Channel Gain Power*
σ_W	*Noise Power*
γ	*Signal-to-Noise Ratio*
$\hat{\gamma}$	*Estimated Signal-to-Noise Ratio*
N	*Number of subcarriers*
n	*Number of samples*
A_i	*Symbol*
p_i	*Probability*
Θ_i	*Vector of Parameters*
ζ	*Sample Moments*

1 INTRODUCTION

This chapter presents the state-of-the-art of the topic in study; the main contributions obtained by us as well as the organization of this work.

1.1 OVERVIEW

The fifth generation (5G) of mobile communications has attracted the attention of several researchers, primarily owing to its capability to deliver notable advancements in speed, latency, and reliability in contrast to earlier generations. In the 5G systems, estimation is a fundamental step as it is directly related to performance optimization, dynamic adaptation, improvement of spectral efficiency and support for new features.

Studies have been presented in the literature concerning estimation. Many of them aim to estimate the channel parameters or the signal-to-noise ratio (SNR), and are usually obtained using the maximum likelihood (ML) or moments (MoM) methods. ML-based estimators are proposed in [1] and [2] respectively, for the parameters of the α-η-μ and α-κ-μ distributions. The estimators derived in the mentioned works are validated by computational simulations. In [3], a simple and closed-form expression for a maximum a posteriori (MAP)-based estimator for the parameter m of the Nakagami-m distribution is presented. MoM-based estimators are presented for the parameters of the κ-μ, η-μ [4] and α-μ [5] distributions. An asymptotically efficient moment-based estimator for the parameter κ of the κ-μ distribution is also presented in [6] and compared with [4]. In [7], a new and exact expression is presented in order to estimate the SNR under Nakagami-m channels for M- and Θ-ary quadrature amplitude modulations (QAM). Expressions to evaluate the variance and mean of the estimates are also derived by the authors in the mentioned work. In [8], new expressions are presented for the SNR estimation and for the mean, variance, and normalized mean square error (NMSE) of the estimates. The authors in [8] uses MoM for SNR estimation, under η-μ and κ-μ fading channels.

In this work, ML and MoM-based estimators are presented for the α-$\mathcal{F}$ composite fading distribution. In our study, ML is considered as it is optimal and is used by us in order to obtain estimates for the parameters of the distribution under analysis. In turn, MoM is considered to estimate the SNR for orthogonal frequency division multiplexing (OFDM) signals. This occur because the likelihood function for this task is intricate and thus, MoM is an alternative to ML technique. It should be mentioned that MoM provides good estimates and is simple, when compared to ML. However, it has the disadvantage of requiring a large number of samples to converge to the real value. In our wok, the α-$\mathcal{F}$ channel model is adopted, that has been extensively supported by experimental results in the technical literature and is written in terms of physical parameters. In addition, the α-$\mathcal{F}$ jointly considers the multipath fading, shadowing and the non-linearity of the propagation medium, that makes this distribution able to model realistic environments. It should be mentioned that the α-$\mathcal{F}$ distribution is generalist. Thus, many works previously presented in the technical literature can be easily found as particular cases of the studies carried out by us (i.e, the α-$\mathcal{F}$

fading model can be used for more general channel characterization if compared to other works). Thus, our study marks a significant contribution to the existing body of knowledge.

1.2 CONTRIBUTIONS OF THIS WORK

This is the first work where estimators for the α-$\mathcal{F}$ distribution are presented. Our work open new fronts for further investigations, since we provide analytical and simulation results to reveal how the channel parameters and SNR range impact the estimator performance. The main contributions of this studies are:

- A ML-based estimator is proposed for the α-$\mathcal{F}$ distribution and validated by means of computational simulations.
- A simple and tractable expression for the SNR estimation is derived, without the need of training sequences, based on the statistical moments of the received signal. This made the estimator proposed by us appropriate for practical applications.

1.3 ORGANIZATION OF THIS WORK

The remaining of the work is organized as follows. Chapter 2 describes the system, channel and SNR models adopted. Estimators for the fading parameters and SNR under α-$\mathcal{F}$ channels are presented in Chapter 3. Chapter 4 shows the numerical results and discussions. Chapter 5 brings the conclusions of the study.

2 SYSTEM, CHANNEL AND SNR MODELS

In this chapter, the system, channel and SNR models considered in our work are presented.

2.1 SYSTEM AND CHANNELS MODELS

The received signal $r[n]$ in the discrete domain, considering a communication system subject to fading and noise, can be written as [9]

$$r[n] = h[n]s[n] + w[n], \tag{2.1}$$

in which $h[n]$ is a random variable (RV) that characterizes the fading and [10]

$$s[n] = \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} x[l] e^{j\frac{2\pi n l}{N}} \tag{2.2}$$

is the OFDM transmitted signal, where N is the number of subcarriers and $x[l]$ is a symbol A_i, from a M-ary QAM constellation, with probability p_i. In turn, $w[n]$ is the additive white Gaussian noise with zero mean and variance $2\sigma_W^2$.

The PDF of $|s[n]|$ is characterized by the Rayleigh distribution with zero mean and unit variance, written as [10]

$$f_{|s[n]|}(s) = 2se^{-s^2}, \tag{2.3}$$

due to the fact that $s[n]$ is a Gaussian random variable with unitary mean and variance σ.

In turn, the fading is modeled in this work by the α-$\mathcal{F}$ distribution, whose PDF is given by [11, Eq. (1)]

$$f_{|h[n]|}(h) = \frac{\alpha}{\mathrm{B}(\mu, m_s)} \left(\frac{\hat{r}^{\alpha}}{\Psi}\right)^{m_s} h^{\alpha\mu-1} \left(h^{\alpha} + \frac{\hat{r}^{\alpha}}{\Psi}\right)^{-(\mu+m_s)}, \tag{2.4}$$

in which $\Psi = \mu/(m-1)$, $\hat{r} = \sqrt[\alpha]{\mathbb{E}[H_{\mathrm{f}}^{\alpha}]}$ denotes the α-root mean value, α characterizes the non-linearity of the propagation medium, μ represents the number of multipath clusters, m is the shadowing parameter and $\mathrm{B}(\cdot,\cdot)$ is the Beta function [12, Eq. (06.18.02.0001.01)]. From the α-$\mathcal{F}$ model, a lot of models presented in the literature can be encompassed. In fact, by properly selecting the fading parameters α, m_s and μ, the models presented in Table 2.1 can be obtained. Fig. 2.1 presents envelope PDF curves for the special cases of the α-$\mathcal{F}$ fading model.

Table 2.1: Special cases.

Fading Models	Parameters
Fisher-Snedecor	$\alpha = 2$, $\mu = \mu$, $m_s = m_s$
α-μ	$\alpha = \alpha$, $\mu = \mu$, $m_s \to \infty$
Weibull	$\alpha = \alpha$, $\mu = 1$, $m_s \to \infty$
Nakagami-m	$\alpha = 2$, $\mu = m$, $m_s \to \infty$
Rayleigh	$\alpha = 2$, $\mu = 1$, $m_s \to \infty$
One-Gaussian	$\alpha = 2$, $\mu = 0.5$, $m_s \to \infty$

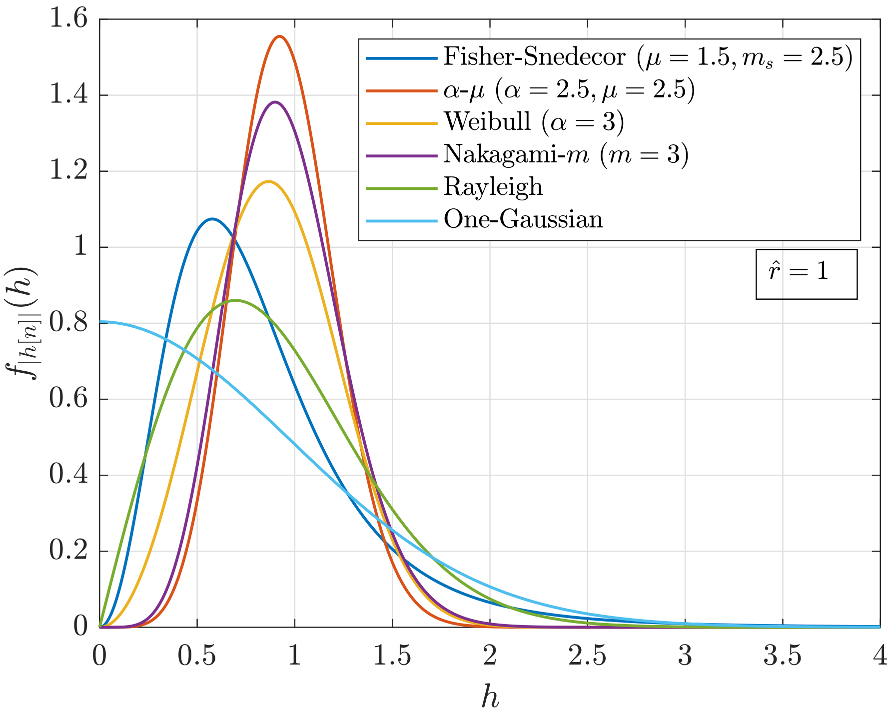


Figure 2.1: Envelope PDF curves for the special cases of the α-$\mathcal{F}$ fading model.

2.2 SNR RECEIVED

The SNR, in the received signal model presented in (2.1), is defined as the ratio between the signal power and the noise power. Mathematically,

$$\text{SNR} = \frac{P_{\text{signal}}}{P_{\text{noise}}}, \tag{2.5}$$

in which

$$P_{\text{noise}} = 2\sigma_W^2, \tag{2.6}$$

and

$$P_{\text{signal}} = \mathbb{E}[|h[n]|^2]\mathbb{E}[|s[n]|^2], \tag{2.7}$$

with $\mathbb{E}[|h[n]|^2] = \sigma_H^2$ and $\mathbb{E}[|s[n]|^2] = \sum_{i=1}^{M} |A_i|^2 p_i$.

Using (2.6) and (2.7) in (2.5), it follows that the SNR is given by

$$\text{SNR} = \frac{\sigma_H^2}{2\sigma_W^2} \sum_{i=0}^{M} |A_i|^2 p_i. \tag{2.8}$$

For a normalized constellation and equiprobable symbols, it follows that $\sum_{i=0}^{M} |A_i|^2 p_i = 1$ and thus,

$$\text{SNR} = \frac{\sigma_H^2}{2\sigma_W^2} = \frac{\gamma}{2}. \tag{2.9}$$

Note in (2.9) that $\gamma = \sigma_H^2/\sigma_W^2$ is the SNR parameter to be estimated. The estimate is denoted by $\hat{\gamma}$ in our work and can be calculated from a function f(γ), defined as the ratio between the square of the second moment and the fourth moment of the received signal.

3 MAXIMUM-LIKELIHOOD AND MOMENT-BASED ESTIMATORS FOR THE α-$\mathcal{F}$ COMPOSITE DISTRIBUTION

ML and MoM-based SNR estimators under α-$\mathcal{F}$ fading channels are presented in this chapter, in which exact and algebraically simple expressions are presented.

3.1 MAXIMUM-LIKELIHOOD ESTIMATOR FOR THE α-$\mathcal{F}$ COMPOSITE DISTRIBUTION

Assuming H_1, H_2, H_3, $\cdots$, H_{N_s} as independent and identically distributed (i.i.d.) RVs, the α-$\mathcal{F}$ joint PDF (JPDF) can be written as the N_s-fold product of α-$\mathcal{F}$ distributions, i.e,

$$f_{\mathbf{H}}(h_1, h_2, \cdots, h_{N_s}; \mathbf{\Theta}) = \prod_{i=1}^{N_s} \frac{\alpha}{\mathrm{B}(\mu, m_s)} \left(\frac{\hat{r}^{\alpha}}{\Psi}\right)^{m_s} h_i^{\alpha\mu-1} \left(h_i^{\alpha} + \frac{\hat{r}^{\alpha}}{\Psi}\right)^{-(\mu+m_s)}. \tag{3.1}$$

In our problem, the ML technique is applied of the α-$\mathcal{F}$ distribution in order to find the estimate for the parameters $\hat{\Theta} = [\hat{\alpha}, \hat{\mu}, \hat{m_s}]$. For this, we can maximize (3.1) or the log-likelihood function $L(\mathbf{h}, \mathbf{\Theta})$, given by

$$L(\mathbf{h}, \mathbf{\Theta}) = \sum_{i=1}^{N_s} \ln[f_{\mathbf{H}}(h_i, \mathbf{\Theta})]. \tag{3.2}$$

In (3.2), $\mathbf{H} = [H_1\, H_2\, H_3 \cdots H_{N_s}]$ is a set of N_s i.i.d. RVs that represent the samples from the α-$\mathcal{F}$ distribution and $\mathbf{\Theta} = [\Theta_1\, \Theta_2\, \Theta_3 \cdots \Theta_{N_s}]$ is the vector of parameters of the mentioned distribution. Thus, considering that the marginal PDF is given by (2.4), it is possible to write (3.2) as

$$\begin{aligned} L(\mathbf{h}, \mathbf{\Theta}) = {} & N_s \ln\left[\frac{\alpha}{\mathrm{B}(\mu, m_s)}\right] + N_s m_s \ln\left[\frac{\hat{r}^{\alpha}}{\Psi}\right] \\ & + (\alpha\mu - 1)\sum_{i=1}^{N_s} \ln[h_i] - (\mu + m_s)\sum_{i=1}^{N_s} \ln\left[h_i^{\alpha} + \frac{\hat{r}^{\alpha}}{\Psi}\right]. \end{aligned} \tag{3.3}$$

From (3.3), it is possible to estimate the parameters of the α-$\mathcal{F}$ distribution taking the derivatives of the mentioned equation in respect to α, μ and m_s, and equals them zero simultaneously. Mathematically,

$$\frac{\partial L(\mathbf{h}, \mathbf{\Theta})}{\partial \Theta_k} = 0. \tag{3.4}$$

Proceeding with some mathematical simplifications, it follows that

$$\begin{aligned}\frac{\partial L(\mathbf{h},\boldsymbol{\Theta})}{\partial\alpha} &= \frac{N_s}{\alpha}+N_s m_s\ln[\hat{r}]+\mu\sum_{i=1}^{N_s}\ln[h_i]\\ &\quad-(\mu+m_s)\sum_{i=1}^{N_s}\frac{h_i^{\alpha}\ln[h_i]}{\left[h_i^{\alpha}+\frac{\hat{r}^{\alpha}}{\Psi}\right]}-(\mu+m_s)\sum_{i=1}^{N_s}\frac{\hat{r}^{\alpha}\ln[\hat{r}]}{\Psi\left[h_i^{\alpha}+\frac{\hat{r}^{\alpha}}{\Psi}\right]},\end{aligned} \tag{3.5}$$

$$\begin{aligned}\frac{\partial L(\mathbf{h},\boldsymbol{\Theta})}{\partial\mu} &= -\frac{N_s\mathrm{B}'(\mu,m_s)}{\mathrm{B}(\mu,m_s)}-\frac{N_s m_s}{\mu}+\alpha\sum_{i=1}^{N_s}\ln[h_i]\\ &\quad-\sum_{i=1}^{N_s}\ln\left[h_i^{\alpha}+\frac{\hat{r}^{\alpha}}{\Psi}\right]+(\mu+m_s)\sum_{i=1}^{N_s}\frac{\hat{r}^{\alpha}(m_s-1)}{\mu^2\left[h_i^{\alpha}+\frac{\hat{r}^{\alpha}}{\Psi}\right]}\end{aligned} \tag{3.6}$$

and

$$\begin{aligned}\frac{\partial L(\mathbf{h},\boldsymbol{\Theta})}{\partial m_s} &= -\frac{N_s\mathrm{B}'(\mu,m_s)}{\mathrm{B}(\mu,m_s)}+N_s\ln\left[\frac{\hat{r}^{\alpha}}{\Psi}\right]+\frac{N_s m_s}{(m_s-1)}\\ &\quad-\sum_{i=1}^{N_s}\ln\left[h_i^{\alpha}+\frac{\hat{r}^{\alpha}}{\Psi}\right]-(\mu+m_s)\sum_{i=1}^{N_s}\frac{\hat{r}^{\alpha}}{\mu\left[h_i^{\alpha}+\frac{\hat{\gamma}^{\alpha}}{\Psi}\right]},\end{aligned} \tag{3.7}$$

with $\mathrm{B}'(\cdot,\cdot)$ being the derivative of $\mathrm{B}(\cdot,\cdot)$.

It should be mentioned that (3.5), (3.6) and (3.7) are contributions of this article. Note that (3.4) is solved in order to find the parameter values that together maximize (3.3). For this task, the Matlab software is used by us.

3.2 MOMENTS-BASED SNR ESTIMATOR FOR THE α-$\mathcal{F}$ COMPOSITE DISTRIBUTION

Firstly in this section, the k-th moment of $|r[n]|$ is calculated and, in sequence, an exact and algebraically simple expression for the SNR estimator in α-$\mathcal{F}$ fading channels for OFDM signals is derived, without the need of training sequences.

3.2.1 Higher-Order Moments of $|r[n]|$

The PDF of $|r[n]|$ conditioned on $|h[n]|$ and $|s[n]|$ is a well know problem. This is similar of calculating the PDF of a RV which has Rice distribution, i.e,

$$f_{|r[n]|}(r\big|h,s)=\frac{r}{\sigma_W^2}e^{-\left(\frac{r^2+h^2s^2}{2\sigma_W^2}\right)}\mathrm{I}_0\left(\frac{rhs}{\sigma_W^2}\right), \tag{3.8}$$

in which $\mathrm{I}_0(\cdot)$ represents the modified Bessel function of zero order. The PDF $f_{|r[n]|}(r|h)$ can be derived by taking the mean of (3.8) with respect to (2.3), as

$$f_{|r[n]|}(r|h) = \frac{r}{\sigma_W^2} \exp\left(-\frac{r^2}{2\sigma_W^2}\right) \int_0^\infty e^{-v\left(1+\frac{h^2}{2\sigma_W^2}\right)} \mathrm{I}_0\left(\frac{rh}{\sigma_W^2}\sqrt{v}\right) \mathrm{d}v. \tag{3.9}$$

The k-th moment of $|r[n]|$ conditioned on the channel gain, denoted by $\mathbb{E}\left[|r[n]|^k \big| |h[n]|\right]$, can be calculated from (3.9) as

$$\mathbb{E}\left[|r[n]|^k \big| |h[n]|\right] = \int_0^\infty r^k f_{r[n]}(r|h)\mathrm{d}r. \tag{3.10}$$

Replacing (3.9) in (3.10) and using [13], it follows that

$$\mathbb{E}\left[|r[n]|^k \big| |h[n]|\right] = 2^{\frac{k}{2}} \left(\sigma_W^2\right)^{\frac{k}{2}} \Gamma\left(\frac{k}{2}+1\right) \int_0^\infty e^{-v\left(1+\frac{h^2}{2\sigma_W^2}\right)} {}_1\mathrm{F}_1\left[\frac{k}{2}+1, 1; \frac{h^2}{2\sigma_W^2}v\right] \mathrm{d}v. \tag{3.11}$$

Finally,

$$\mathbb{E}\left[|r[n]|^k\right] = M_k = \int_0^\infty \mathbb{E}\left[|r[n]|^k \big| |h[n]|\right] f_{|h[n]|}(h)\mathrm{d}h. \tag{3.12}$$

Using (2.4) and proceeding with some simplifications,

$$\begin{aligned} M_k = {} & 2^{\frac{k}{2}} (\sigma_W^2)^{\frac{k}{2}} \Gamma\left(\frac{k}{2}+1\right) \frac{\alpha}{\mathrm{B}(\mu, m_s)} \left(\frac{\hat{r}^\alpha}{\Psi}\right)^{m_s} \\ & \times \int_0^\infty e^{-v} \left[\int_0^\infty e^{-\frac{h^2}{2\sigma_W^2}v} \frac{h^{\alpha\mu-1}}{\left(h^\alpha + \frac{\hat{r}^\alpha}{\Psi}\right)^{(\mu+m_s)}} {}_1\mathrm{F}_1\left(\frac{k}{2}+1, 1; \frac{h^2}{2\sigma_W^2}v\right) \mathrm{d}h\right] \mathrm{d}v \end{aligned} \tag{3.13}$$

Using the fact the

$${}_1\mathrm{F}_1\left(\frac{k}{2}+1, 1; \frac{h^2}{2\sigma_W^2}v\right) = e^{\frac{h^2}{2\sigma_W^2}v} \mathrm{L}_{k-1}\left(-\frac{h^2}{2\sigma_W^2}v\right), \tag{3.14}$$

in which [13]

$$\mathrm{L}_k(x) = \frac{1}{k!} e^x \frac{d^k}{dx^k}(x^k e^{-x}) \tag{3.15}$$

is the Laguerre polynomial of order $k-1$; and knowing [13, Eq. (3.241.4)] and [12, id (06.05.02.0001.01)], then it is possible to deduce the moments of order 2, 4, 6 and 8 by replacing (3.14) in (3.13). Proceeding with some simplifications, the mentioned moments can be written in terms of the SNR γ as

$$\begin{cases} M_2 = \sigma_W^2(\mathcal{A}\gamma + 2) \\ M_4 = \sigma_W^4\left(2\mathcal{B}\gamma^2 + 8\mathcal{A}\gamma + 8\right) \\ M_6 = \sigma_W^6\left(6\mathcal{C}\gamma^3 + 36\mathcal{B}\gamma^2 + 72\mathcal{A}\gamma + 48\right) \\ M_8 = \sigma_W^8\left(24\mathcal{D}\gamma^4 + 192\mathcal{C}\gamma^3 + 576\mathcal{B}\gamma^2 + 768\mathcal{A}\gamma + 24\right) \end{cases}, \tag{3.16}$$

with

$$\begin{cases} \mathcal{A} = \dfrac{\Gamma\left(m_s - \frac{2}{\alpha}\right)\Gamma\left(\mu + \frac{2}{\alpha}\right)}{\Gamma\left(m_s\right)\Gamma\left(\mu\right)}\Psi^{-\frac{2}{\alpha}} \\ \mathcal{B} = \dfrac{\Gamma\left(m_s - \frac{4}{\alpha}\right)\Gamma\left(\mu + \frac{4}{\alpha}\right)}{\Gamma\left(m_s\right)\Gamma\left(\mu\right)}\Psi^{-\frac{4}{\alpha}} \\ \mathcal{C} = \dfrac{\Gamma\left(m_s - \frac{6}{\alpha}\right)\Gamma\left(\mu + \frac{6}{\alpha}\right)}{\Gamma\left(m_s\right)\Gamma\left(\mu\right)}\Psi^{-\frac{6}{\alpha}} \\ \mathcal{D} = \dfrac{\Gamma\left(m_s - \frac{8}{\alpha}\right)\Gamma\left(\mu + \frac{8}{\alpha}\right)}{\Gamma\left(m_s\right)\Gamma\left(\mu\right)}\Psi^{-\frac{8}{\alpha}} \\ \gamma = \dfrac{\sigma_H^2}{\sigma_W^2} \end{cases} . \tag{3.17}$$

Note that the moments presented are theoretical and necessary to calculate the SNR estimator and its variance.

3.2.2 SNR-based Estimator

Let be $f(\gamma)$ a function defined as the ratio between the theoretical second moment squared to the fourth moment, i.e.,

$$f(\gamma) = \frac{M_2^2}{M_4}. \tag{3.18}$$

Using the MoM technique, the theoretical and sample moments can be equated as

$$f(\gamma) = \frac{M_2^2}{M_4} = \frac{\zeta_2^2}{\zeta_4}, \tag{3.19}$$

in which

$$\zeta_k = \frac{1}{N}\sum_{n=1}^{N} |r[n]|^k \tag{3.20}$$

is the k-th sample moment.

From (3.19), the estimated SNR $\hat{\gamma}$ can be calculated as

$$\hat{\gamma} = f^{-1}\left(\frac{\zeta_2^2}{\zeta_4}\right), \tag{3.21}$$

in which $f_\zeta = \zeta_2^2/\zeta_4$ is the ratio between the sample moments of $|r[n]|$. Note that the process of deriving an expression for the SNR estimator from (3.21) consists of finding $f^{-1}(\cdot)$ such that $f^{-1}(f_\zeta)$ exists.

Using the expressions for M_2 and M_4 in (3.17), it follows from (3.18) that

$$f(\gamma) = \frac{(\mathcal{A}\gamma + 2)^2}{2\mathcal{B}\gamma^2 + 8\mathcal{A}\gamma + 8}. \tag{3.22}$$

Hence, the estimate $\hat{\gamma}$ obtained after inverting (3.21) in terms of f_ζ can be written as

$$\hat{\gamma} = \frac{2\mathcal{A} - 4\mathcal{A}f_\zeta + \sqrt{16\mathcal{A}^2 f_\zeta^2 - 8\mathcal{A}^2 f_\zeta - 8\mathcal{B}f_\zeta}}{2\mathcal{B}f_\zeta - \mathcal{A}^2}. \tag{3.23}$$

Note that $\hat{\gamma}$ depends on the statistical characteristics of the channel, which are related to the fading model considered.

3.2.3 Evaluation of the Proposed SNR Estimator

In this subsection, the variance statistical is analyzed for the estimator proposed when the method of moments is applied. This is important given the need to find some parameter that measures its performance. Due to the non-linearity of the SNR estimate, it is practically impossible to find the Crammer-Rao bound for this estimate. In this way, an expression for the variance of the estimator is presented.

Firstly, $\hat{\gamma}$ in (3.23) is written as a function of the sample moments of $|r[n]|$ in the form

$$\begin{aligned}\hat{\gamma} &= -2\mathcal{A}(2g(T_1,T_2) - 1)(2\mathcal{B}g(T_1,T_2) - \mathcal{A}^2)^{-1} \\ &\quad + \sqrt{16\mathcal{A}^2 g^2(T_1,T_2) - 8\mathcal{A}^2 g(T_1,T_2) - 8\mathcal{B}g(T_1,T_2)}(2\mathcal{B}g(T_1,T_2) - \mathcal{A}^2)^{-1}.\end{aligned} \tag{3.24}$$

in which $g(T_1,T_2) = T_1^2/T_2$, such that

$$\mathbb{E}[T_k] = \frac{1}{N}\sum_{n=1}^{N}\mathbb{E}[|r[n]|^{2k}]. \tag{3.25}$$

Using the results presented in [14] and [7, Eq. (44)], the estimated variance may be expressed as

$$\text{Var}[\hat{\gamma}] = \frac{1}{N}z^2(f_M)\left[4f_M + f_M^2\left(\frac{M_8}{M_4^2} - \frac{4M_6}{M_2M_4} - 1\right)\right], \tag{3.26}$$

in which $f_M = M_2^2/M_4$ and $z\,(x)$ is the function obtained deriving (3.24) from $x = g(T_1,T_2)$. After simplifications,

$$\begin{aligned}z\,(x) = (2\mathcal{B}g - \mathcal{A}^2)^{-1}\Big[&4(16\mathcal{A}^2g^2 - 8\mathcal{A}^2g - 8\mathcal{B}g)^{-\frac{1}{2}}(4\mathcal{A}^2g - \mathcal{A}^2 - \mathcal{B}) - 2\mathcal{B}(2\mathcal{B}g - \mathcal{A}^2)^{-2} \\ &(16\mathcal{A}^2g^2 - 8\mathcal{A}^2g - 8\mathcal{B}g) + 4\mathcal{A}\mathcal{B}(2g-1)(2\mathcal{B}g - \mathcal{A}^2)^{-2} - 4\mathcal{A}(2\mathcal{B}g - \mathcal{A}^2)^{-1}\Big].\end{aligned} \tag{3.27}$$

Note from (3.26) that the performance of the estimator becomes better as the number of observed samples N increases.

4 RESULTS

In this chapter, theoretical curves were carried out by using the MATLAB software according to the models described throughout the paper.

4.1 NUMERICAL RESULTS FOR THE ML-BASED ESTIMATOR OF THE α-$\mathcal{F}$ COMPOSITE DISTRIBUTION

In this work, 100 estimations are conducted for each displayed result, with

$$\hat{\Theta}_i = \frac{1}{100}\sum_{k=1}^{100}\left[\hat{\Theta}_k\right]_i, \tag{4.1}$$

in which $\left[\hat{\Theta}_k\right]_i$ is the k-th estimation of the i-th parameter. Furthermore, 95% confidence level is adopted. This confidence level corresponds to approximately 1.96 standard deviations to the right and left of the mean of the Gaussian distribution and is expressed by

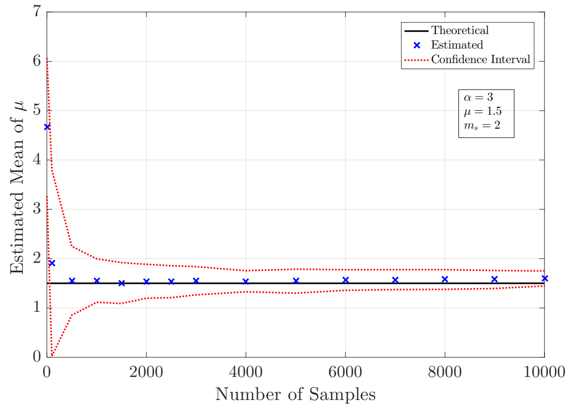
$$\left[\hat{\Theta}_i - 1.96\sqrt{\mathbb{V}\left[\hat{\Theta}_i\right]}, \hat{\Theta}_i + 1.96\sqrt{\mathbb{V}\left[\hat{\Theta}_i\right]}\right]. \tag{4.2}$$

In addition, the variance $\mathbb{V}(\cdot)$ of the estimators is given by

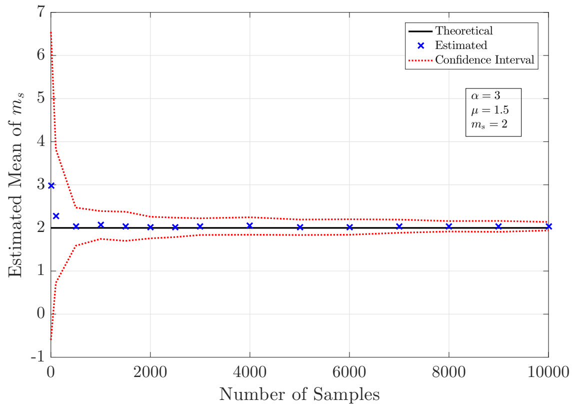
$$\mathbb{V}[\hat{\Theta}_i] = \frac{1}{20}\sum_{k=1}^{20}\left(\left[\hat{\Theta}_k\right]_i - \mathbb{E}[\hat{\Theta}_k]\right)^2. \tag{4.3}$$

Figures 4.1(a), (b) and (c) shown the estimated values of the parameters μ, m_s and α as a function of the number of samples, respectively. In our simulations, we adopted the acceptance-rejection method for the samples generation. An improvement in the precision of the estimators is perceived in Figure 4.1 as the number of samples increases, both in terms of the mean and the confidence interval. It should be mentioned that the values converge quickly using this method, even for a small number of samples.

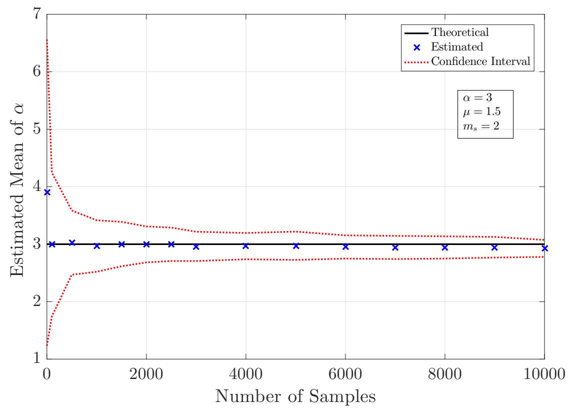
Figure 4.2 illustrates the comparison between the theoretical PDFs generated using the values Θ = [3 1.5 2], Θ = [1 5 3] and Θ = [2 2 1.5]; with the PDFs obtained with the estimated parameter values, for 10000 samples. The corresponding estimated parameters obtained are $\hat{\Theta}$ = [3.1286 1.4429 1.8714], $\hat{\Theta}$ = [1 4.8571 3.0429] and Θ = [1.8857 2.2571 1.5714], for Figure 4.2(a), (b) and (c) respectively. Note that this study is made in order to ascertain the effectiveness in predicting the behavior of PDF envelope. Despite some apparent imprecision's in the estimations, from an analysis of the envelope PDFs resulting, in which the theoretical PDF is obtained using the parameter values adopted in the sample generation, and the estimated PDF is obtained using the values of the jointly estimated parameters; it can be shown that the ML technique is effective in predicting the behavior of the signal envelope in α-$\mathcal{F}$ channels.



(a)



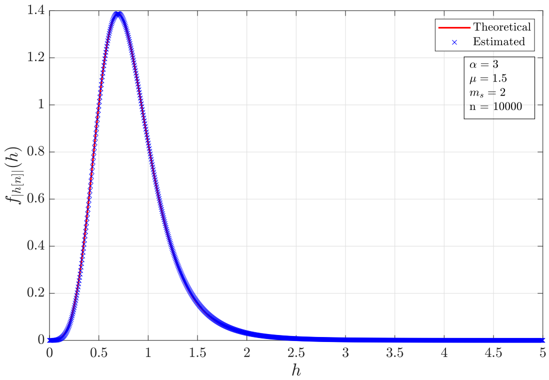
(b)



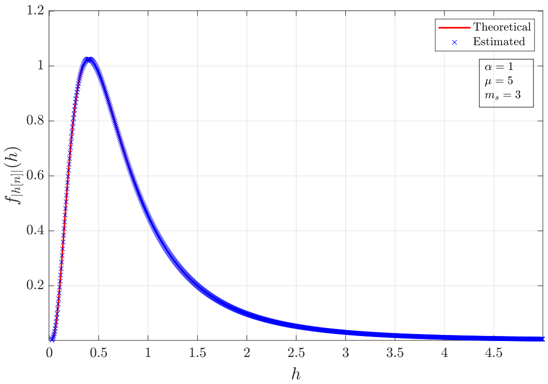
(c)

Figure 4.1: Mean of the estimations of (a) μ, (b) m_s and (c) α as a function of the number of samples.

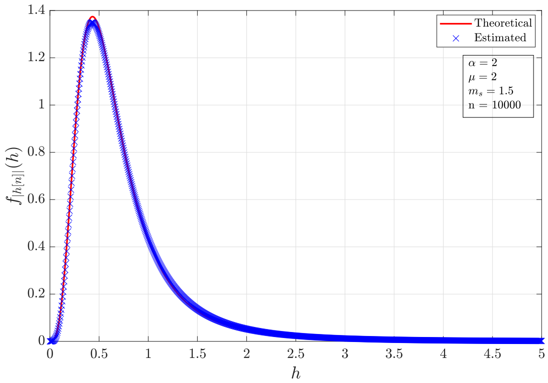
Figure 4.3 shows the estimation results and confidence intervals, compared to the real values of the parameters used to generate the samples, for different values of (a) m_s, (b) μ and (c) α. The results suggest that the MLE estimator performs well for different channel scenarios characterized by the values of the α-$\mathcal{F}$ fading distribution. It can also be seen from the precision of the estimated values and the size of the confidence interval that α is more precise than μ and m_s.



(a)



(b)

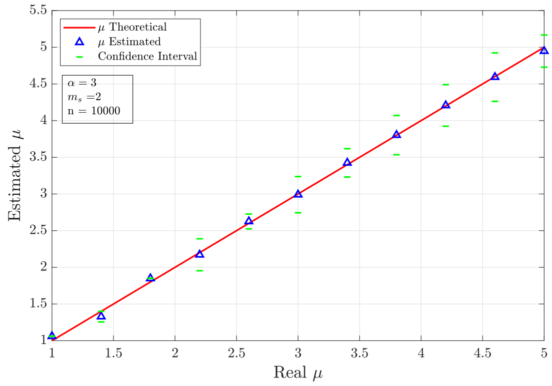


(c)

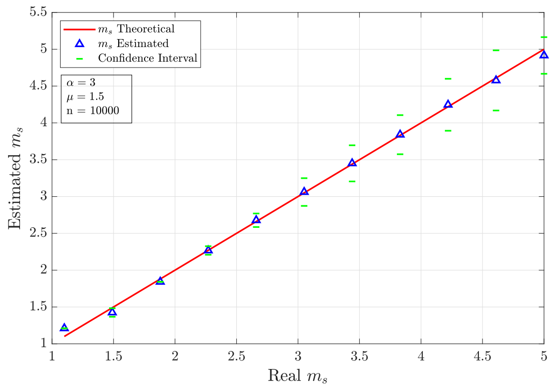
Figure 4.2: Envelope PDFs generated using the theoretical and estimated parameters of the α-$\mathcal{F}$ distribution for (a) Θ = [3 1.5 2], (b) Θ = [1 5 3] and (c) Θ = [2 2 1.5].

4.2 NUMERICAL RESULTS FOR THE MOMENTS-BASED SNR ESTIMATOR OF THE α-$\mathcal{F}$ COMPOSITE DISTRIBUTION

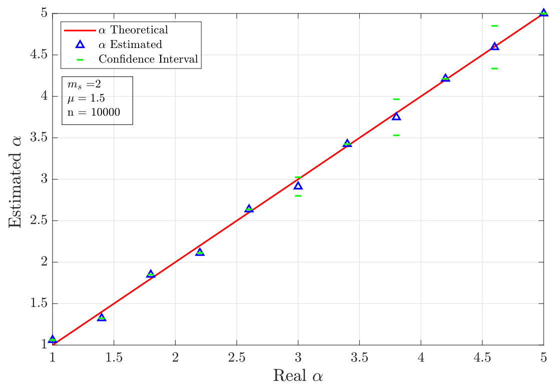
Curves of $f(\gamma)$ as a function of the SNR are presented in Fig. 4.4 for different values of m_s, α and μ. For $\alpha = 2$, the Fisher-Snedecor case is provided as a benchmark. These curves are shown in order to highlight the impact of channel parameters on estimator performance. Note that all curves presented tend



(a)



(b)



(c)

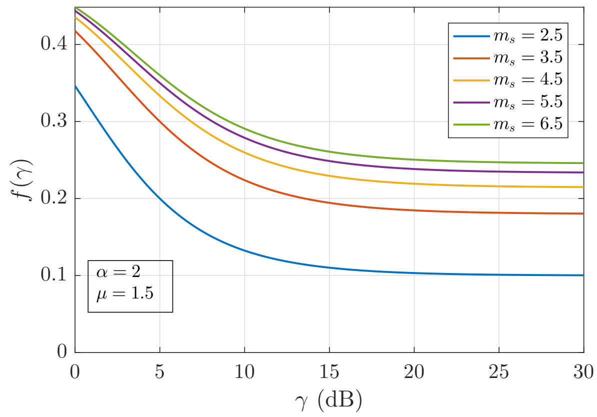
Figure 4.3: Curves of the mean of the estimations of the parameters as a function of the real parameters.

towards a horizontal asymptote as $\gamma \to \infty$, since

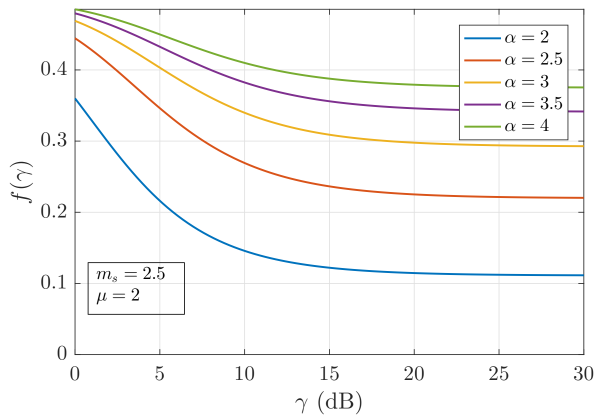
$$\lim_{\gamma \to +\infty} f(\gamma) = \frac{\mathcal{A}^2}{2\mathcal{B}}. \tag{4.4}$$

.

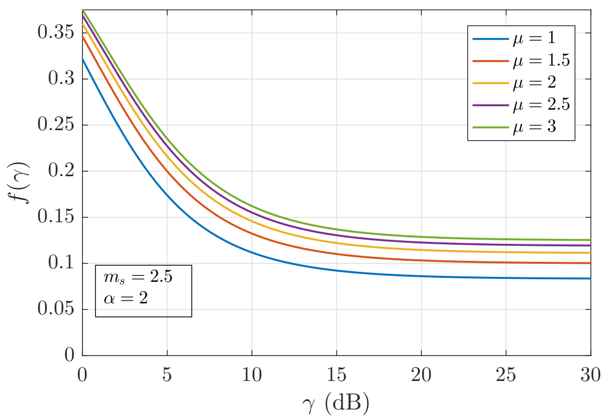
From Fig 4.4, it is noticed that $f(\gamma)$ present different values as m_s, α and μ changes. Furthermore, it appears that the invertibility of $f(\gamma)$ is only guaranteed for SNR values lower than 30 dB. This indicates that our estimator performs well up to this threshold. Furthermore, it should be mentioned that the estimator proposed is valid for weak and strong shadowing, multipath fading and/or non-linearity conditions.



(a)



(b)



(c)

Figure 4.4: Curves of $f(\gamma)$ as a function of the SNR are presented, for different values of (a) m_s, (b) α and (c) μ.

5 CONCLUSIONS

This work advanced the knowledge of estimation under α-$\mathcal{F}$ fading channels. In this study, the maximum-likelihood (ML) and moments-based estimators for the α-$\mathcal{F}$ composite fading distribution were proposed. Firstly, a ML-based estimator was presented for the parameters of the α-$\mathcal{F}$ distribution. Secondly, a new expression for the signal-to-noise ratio (SNR) estimation was derived, based on the statistical moments of the envelope samples of the received signal. Several curves were presented and validated by simulations.

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